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ERRATUM: A LOCAL PROOF OF PETRI'S CONJECTURE AT THE GENERAL CURVE

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Abstract

A generation of symbols asserted for $n \geq 0$ in the proof of Theorem 3.3 of the original paper in fact only holds for $n > 0$, thus undermining the proof of the theorem. A new version of Section 3.5 of the original paper is given, culminating in a corrected proof of Theorem 3.3. The author thanks Deepak Khosla for pointing out the gap in the previous version of the proof.

3.5 Extendable linear systems on curves.

If M denotes a sufficiently small analytic neighborhood of a general point in the moduli space of curves of genus g, with universal curve C/M, there is a stratification of the locus

$$Z_d^r = \left\{ L : L \text{ globally generated, } h^0(L) = r+1 \right\} \subseteq \operatorname{Pic}^d(C/M)$$

such that all strata are smooth and the projection of each to M is submersive with diffeomeorphic fibers. Next consider the induced stratification of the pre-image of Z_d^r under the Abel-Jacobi map

$$\alpha : C^{(d)}/M \to \operatorname{Pic}^d(C/M).$$

By considering the contact locus between this pre-image stratification and the various diagonal loci in $C^{(d)}/M$, one can construct a refinement of the stratification of

$$\alpha^{-1}(Z_d^r) \subseteq C^{(d)}/M$$

such that all strata are smooth and the projection of each to M is submersive with diffeomorphic fibers and having the additional property that, beginning with the initial element (d) of the partially ordered set $\{(d_1, \ldots, d_s)\}$ of all partitions of d, the stratification is compatible with each set

$$\operatorname{diag}_{(d_1,\ldots,d_s)}\left(C^{(d)}/M_g\right) \cap \alpha^{-1}(Z_d^r).$$

Suppose now that C_0 is a compact Riemann surface of genus g of general moduli and that L_0 is a line bundle of degree d on C_0 such that the linear system $\mathbb{P}_0 := \mathbb{P}\left(H^0(L_0)\right)$ is basepoint-free. Let C_β/Δ be a Schiffer variation supported at a finite set $A_0 \subseteq C_0$. Then, by genericity

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of C_0 and the remarks just above, there is a deformation $\mathbb{P}_\Delta \subseteq C_\beta^{(d)}$ over Δ of $\mathbb{P}_0 \subseteq C_0^{(d)}$ for which there exists a trivialization

$$T : \mathbb{P}_\Delta \to \mathbb{P}_0 \times \Delta \tag{30}$$

compatible with each partition locus of d, that is, for each partition $(d_1, \ldots, d_s)$ of d,

$$\begin{aligned} T\left(\operatorname{diag}_{(d_1,\ldots,d_s)}\left(C_\beta^{(d)}\right) \times_{C_\beta^{(d)}} \mathbb{P}_\Delta\right) \\ = \left(\operatorname{diag}_{(d_1,\ldots,d_s)}\left(C_0^{(d)}\right) \times_{C_0^{(d)}} \mathbb{P}_0\right) \times \Delta. \end{aligned} \tag{31}$$

Notice that T is a C^∞-map, and is not in general analytic. However T can be chosen so that, for each $p \in \mathbb{P}_0$, $T^{-1}(\{p\} \times \Delta)$ is a proper analytic subvariety of $\mathbb{P}_\Delta$.

Now the tautological section $\tilde{f}_0$ of $\tilde{L}_0(1) = \mathcal{O}_{\mathbb{P}_0} \boxtimes L_0$ defined in (27) has divisor

$$D_0 \subseteq \mathbb{P}_0 \times C_0.$$

Let

$$D \subseteq \mathbb{P}_\Delta \times_\Delta C_\beta$$

denote the divisor of the tautological section $\tilde{f}$ of

$$\tilde{L}(1) := \mathcal{O}_{\mathbb{P}_\Delta}(1) \boxtimes_\Delta L.$$

Then, by (31), the "product" trivialization

$$(T, F_\beta) : \mathbb{P}_\Delta \times_\Delta C_\beta \to \mathbb{P}_0 \times C_0 \times \Delta$$

is compatible with the trivialization T in (30), that is, for each $p \in \mathbb{P}_0$,

$$(T, F_\beta)^{-1}(\{p\} \times C_0 \times \Delta) = T^{-1}(\{p\} \times \Delta) \times_{\mathbb{P}_\Delta} (\mathbb{P}_\Delta \times_\Delta C_\beta).$$

That is, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{P}_\Delta \times_\Delta C_\beta & \xrightarrow{(T,F_\beta)} & \mathbb{P}_0 \times C_0 \times \Delta \\ \downarrow & & \downarrow \\ \mathbb{P}_\Delta & \xrightarrow{T} & \mathbb{P}_0 \times \Delta \end{array}$$

Furthermore, by (31), we can adjust (T, F_β) "in the C_0-direction" to obtain a trivialization

$$\begin{array}{ccc} \mathbb{P}_\Delta \times_\Delta C_\beta & \xrightarrow{F} & \mathbb{P}_0 \times C_0 \times \Delta \\ \downarrow & & \downarrow \\ \mathbb{P}_\Delta & \xrightarrow{T} & \mathbb{P}_0 \times \Delta \end{array}$$

which maintains the property

$$F^{-1}(\{p\} \times C_0 \times \Delta) = T^{-1}(\{p\} \times \Delta) \times_{\mathbb{P}_\Delta} (\mathbb{P}_\Delta \times_\Delta C_\beta). \tag{32}$$

and achieves in addition that

$$F^{-1}(D_0 \times \Delta) = D. \tag{33}$$

Finally, we can choose the adjustments to be holomorphic in the C_0-direction in a small neighborhood of $\mathbb{P}_\Delta \times A_0 \times \Delta$.

Thus referring to Lemma 2.7 there is a C^∞-vector field

$$\gamma = \sum\nolimits_{n>0} \gamma_n t^n$$

on $\mathbb{P}_0 \times C_0 \times \Delta$ of type $(1,0)$ such that

1) each γ_n annihilates functions pulled back from $\mathbb{P}_0$, that is, it is an $\mathcal{O}_{\mathbb{P}_0}$-linear operator,

2) for each n and each $p \in \mathbb{P}_0$,

$$\gamma_n|_{\{p\}\times C_0}$$

is meromorphic on a neighborhood of $\{p\} \times A_0$,

3) given a function

$$g = \sum\nolimits_{k=0}^{\infty} g_k t^k : \mathbb{P}_0 \times C_0 \times \Delta \to \mathbb{C}$$

with each g_k a C^∞-function on (an open set in) $\mathbb{P}_0 \times C_0$ and any point $p \in \mathbb{P}_0$,

$$g \circ F|_{T^{-1}(\{p\}\times\Delta)\times_{\mathbb{P}_\Delta}(\mathbb{P}_\Delta \times_\Delta C_\beta)}$$

is holomorphic if and only if

$$\left[\overline{\partial_0}, e^{L_{-\gamma}}\right](g)\Big|_{\{p\}\times C_0\times\Delta} = 0.$$

Again, following Lemma 3.2, there is a trivalization

$$\begin{array}{ccc} \tilde{L}(1)^\vee & \xrightarrow{\tilde{F}} & \tilde{L}_0(1)^\vee \times \Delta \\ \downarrow & & \downarrow \\ \mathbb{P}_\Delta \times_\Delta C_\beta & \xrightarrow{F} & \mathbb{P}_0 \times C_0 \times \Delta \end{array}$$

of $\tilde{L}(1)$ and a lifting $\tilde{\gamma}$ of γ such that, for the tautological sections $\tilde{f}_0$ and $\tilde{f}$ defined earlier in this section,

$$\tilde{f} = \tilde{F} \circ \tilde{f}_0.$$

Thus, for each $p \in \mathbb{P}_0$,

$$\left[\overline{\partial_0}, e^{L_{-\tilde{\gamma}}}\right]\left(\tilde{f}_0\right)\Big|_{\{p\}\times C_0\times\Delta} = 0. \tag{34}$$

Let

$$\mathfrak{D}_n^{\mathbb{P}_0}\left(\tilde{L}_0(1)\right) \subseteq \mathfrak{D}_n\left(\tilde{L}_0(1)\right)$$

denotes the subsheaf of $\mathcal{O}_{\mathbb{P}_0}$-linear operators. Then

$$\left[\overline{\partial_0}, e^{L_{-\tilde{\gamma}}}\right]$$

is a $\overline{\partial_0}$-closed element of

$$\sum\nolimits_{n>0}^{\infty} H^1\left(\mathfrak{D}_n^{\mathbb{P}_0}\left(\tilde{L}_0(1)\right)\right) t^n.$$

Now, referring to (29), we need to analyze

$$\rho_* \left[\overline{\partial_0}, e^{L_{-\tilde{\gamma}}}\right] \in \sum\nolimits_{n>0} H^1\left(\mathfrak{D}'_n\right) t^n$$
$$= \sum\nolimits_{n>0} H^1\left(\mathfrak{D}_n\left(L_0\right)\right) \otimes \operatorname{End}\left(H^0\left(L_0\right)\right) t^n.$$

In fact, by construction, this element lies in the image of

$$\sum\nolimits_{n>0}^{\infty} H^1\left(\rho_* \mathfrak{D}_n^{\mathbb{P}_0}\left(\tilde{L}\left(1\right)\right)\right) t^n$$
$$= \sum\nolimits_{n>0} H^1\left(\mathfrak{D}_n\left(L_0\right)\right) \otimes \mathbb{C} \cdot (id) \cdot t^n$$
$$\subseteq \sum\nolimits_{n>0} H^1\left(\mathfrak{D}_n\left(L_0\right)\right) \otimes \operatorname{End}\left(H^0\left(L_0\right)\right) t^n.$$

Now

$$H^1\left(\tilde{L}_0\left(1\right)\right) = \operatorname{Hom}\left(H^0\left(L_0\right), H^1\left(L_0\right)\right).$$

But by (34), the image of

$$\left.\left\{\left[\overline{\partial_0}, e^{L_{-\tilde{\gamma}}}\right]\left(\tilde{f}_0\right)\right\}\right|_{\{p\}\times C_0 \times \Delta} \in \sum\nolimits_{n>0}^{\infty} H^1\left(L_0\right) \cdot t^n.$$

is zero for each $p \in \mathbb{P}_0$. Thus

$$\rho_* \left[\overline{\partial_0}, e^{L_{-\tilde{\gamma}}}\right]\left(\rho_* \tilde{f}_0\right) = 0 \in \sum\nolimits_{n>0} \operatorname{Hom}\left(H^0\left(L_0\right), H^1\left(L_0\right)\right) t^n. \tag{35}$$

Theorem 3.3. *Suppose X_0 is a curve of genus g of general moduli. Suppose further that, by varying the choice of β in the Schiffer-type deformation in Section* 3.3, *the coefficients to t^{n+1} in all expressions*

$$\left[\overline{\partial}, e^{-L_\beta}\right]$$

generate $H^1\left(S^{n+1}\left(T_{X_0}\right)\right)$ for each $n \geq 0$. (For example we allow the divisor $A_0 \subseteq X_0$ to move.) Then the maps

$$\mu^{n+1} : H^1\left(S^{n+1} T_{X_0}\right) \to \frac{\operatorname{Hom}\left(H^0\left(L_0\right), H^1\left(L_0\right)\right)}{\operatorname{image} \tilde{\mu}^n}$$

are zero for all $n \geq 0$.

Proof. Let

$$\rho_* \left[\overline{\partial_0}, e^{L_{-\tilde{\gamma}}}\right]_{n+1}.$$

denote the coefficient of t^{n+1} in $\rho_* \left[\overline{\partial_0}, e^{L_{-\tilde{\gamma}}}\right]$. Referring to (29) and the fact the the operators take values in the sheaf $\mathfrak{D}_n^{\mathbb{P}_0}\left(\tilde{L}_0\left(1\right)\right)$, we have that

(36)

$$\operatorname{symbol}\left(\left(\rho_* \left[\overline{\partial_0}, e^{L_{-\tilde{\gamma}}}\right]\right)_{n+1}\right)$$
$$= \left(\overline{\partial}\beta_1^{n+1} \otimes 1\right) \oplus 0 \in S^{n+1}\left(T_{X_0}\right) \oplus \left(S^n\left(T_{X_0}\right) \otimes \operatorname{End}^0\left(H^0\left(L_0\right)\right)\right)$$

where
$$\beta = \sum\nolimits_{j>0} \beta_j t^j .$$
By (36) and the hypothesis that the elements $\overline{\partial}\beta_1^{n+1}$ generate $H^1\left(S^{n+1}T_{X_0}\right)$, we have that, by varying β, the elements
$$\text{symbol}\left(\rho_* \left[\overline{\partial_0}, e^{L-\tilde{\gamma}}\right]_{n+1}\right)$$
generate
$$S^{n+1}\left(T_{X_0}\right)$$
for each $n \geq 0$.

Thus, by (29) and (35), the map $\tilde{\nu}^{n+1}$ given by
$$\begin{array}{rcl} H^1\left(\mathfrak{D}_{n+1}\left(L_0\right)\right) & \rightarrow & \dfrac{\operatorname{Hom}\left(H^0\left(L_0\right), H^1\left(L_0\right)\right)}{\text{image}\left(\tilde{\nu}^n\right)} \\ D & \mapsto & D\left(\tilde{f}_0\right) \end{array}$$
is zero for all $n \geq 0$. q.e.d.

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